Metrizability of Furst. boundary

Let G be a Polish group

- A G-flow, X, is a cpt Haus space w/ $G \curvearrowright X$ ctsly,
- X is min'l if $\forall x \in X$, $G \cdot x$ is dense in X

For any G, there is a (unique up to isom.) universal min'l G-flow, $\mathcal{M}(G)$, i.e., $\mathcal{M}(G)$ is a min'l G-flow and $\forall$ min'l G-flow X $\exists$ cts, equiv, map

$$\mathcal{M}(G) \twoheadrightarrow X$$

- For G non-cpt, locally cpt, $\mathcal{M}(G)$ is non-metrizable and $|\mathcal{M}(G)| > |\mathbb{R}|$. [Veech]

Defn: G is extremely amenable if every G-flow X has a fixed point ($\exists\, x \in X$ s.t. $\forall g \in G$ $g \cdot x = x$)

$$\iff |\mathcal{M}(G)| = 1$$

Ex: $\mathrm{Aut}(\mathbb{Q}, \leq)$ [Pestov]

$\mathrm{Aut}([0,1], \lambda)$ = meas.-pres. transformations of std. Lebesgue space [Giordano-Pestov]

Defn: G has metrizable u.m.f. if $\mathcal{M}(G)$ is metrizable

Ex: S_∞, $\mathcal{M}(S_\infty) = S_\infty \curvearrowright LO(\mathbb{N})$ [Glasner-Weiss]

Homeo $(2^{\mathbb{N}})$ [Kechris - Pestov - Todorcevic]

Defn: A <u>convex-G-flow</u> is $G \curvearrowright K$ by affine transf. and K is a cpt, convex subset of a LCTVS

<u>Ex:</u> $G \curvearrowright X$ a flow

Let $\mathrm{Prob}(X) = \{$all Borel prob. measures on $X\}$ with weak-$*$ topology

$\rightsquigarrow G \curvearrowright \mathrm{Prob}(X)$ by $g \cdot \mu(A) = \mu(gA)$ is a convex-G-flow

Defn: G is <u>amenable</u> if every convex-G-flow has a fixed point $\Leftrightarrow$ every G-flow has an invariant Borel prob. measure

Defn: A G-flow X is <u>strongly proximal</u> if $\forall \mu \in \mathrm{Prob}(X)$ $\exists$ net $g_i \in G$, $g_i \cdot \mu \longrightarrow$ a Dirac mass.

⋆ Defn ⋆ the <u>Furstenberg boundary</u> of G is the universal min'l strongly proximal flow, denoted $\Pi_{sp}(G)$

($\Pi_{sp}(G)$ is a min'l s.p. G-flow + it quotients onto any min'l s.p. G-flow)

Fact: G is amenable $\Leftarrow\Rightarrow$ $|\Pi_{sp}(G)| = 1$

So First. bdry is like u.m.f. but for amenability.

Rmk: A convex-G-flow is irreducible if it contains no proper convex-G-subflow.

Prob $(\Pi_{sp}(G))$ = universal irreducible convex-G-flow.

Defn: G has metrizable F.B. if $\Pi_{sp}(G)$ is metrizable

Note: amenable $\Rightarrow$ metrizable F.B.

metrizable u.m.f. $\not\Rightarrow$

Fact: G discrete group. Then: G amenable $\Updownarrow$ G has metr. F.B.

Characterizations of "dynamical conditions" via combinatorics:

Extreme Amen $\Leftarrow\Rightarrow$ Ramsey prop. [Kechris-Pestov-Todorcevic] metr. u.m.f. $\Longleftrightarrow$ finite Ramsey degrees [Zucker]

amenability $\Longleftrightarrow$ convex Ramsey property [Moore]

metr. F.B. $\Longleftrightarrow$?

<u>Thm</u> (Bartosova) G a top. gp.

G is extremely amen.

$\Longleftrightarrow$ $\forall\, \mathcal{U} \ni 1_G$ open, S, T syndetic, open,

$$\mathcal{U}S \cap \mathcal{U}T \neq \emptyset$$

(A version of KPT correspondence)

<u>Defn</u>: $S \subseteq G$ is <u>syndetic</u> if $\exists\, F \subset G$ finite w/ $FS = G$.

<u>Defn</u>: $S \subseteq G$ is <u>convexly syndetic</u> if $\forall\, \varepsilon > 0$ $\exists\, F \subseteq G$ fin. s.t. for any finitely spted prob. meas. ρ on G, $\exists\, f \in F$ s.t.

$$\rho(fS) > 1 - \varepsilon.$$

(convexly syndetic $\Rightarrow$ syndetic)

For $\mathcal{U} \ni 1_G$ open, the <u>convex Ramsey deg. of $\mathcal{U}$ is $\geq k$</u> if $\exists\, (S_1, \ldots, S_k)$ open convexly syndetic ; for $i \neq j$ $\mathcal{U}S_i \cap \mathcal{U}S_j = \emptyset$.

let $CRdeg(\mathcal{U}) = \max\{k :$ convex Ramsey degree of $\mathcal{U}$ is $\geq k\}$

or ∞ if convex Ramsey degree of $\mathcal{U}$ is $\geq k \ \forall k \in \mathbb{N}$.

<u>Thm</u> (Dovnat - I - Shinko) let G be Polish. G has metr. Furstenberg bdry $\iff$ for all $\mathcal{U} \ni 1_G$ open, $CRdeg(\mathcal{U})$ is finite.

<u>Rmk</u>: thm gives combinatorial conditions ("finite convex Ramsey degrees of structures") when G is non-archimedean.

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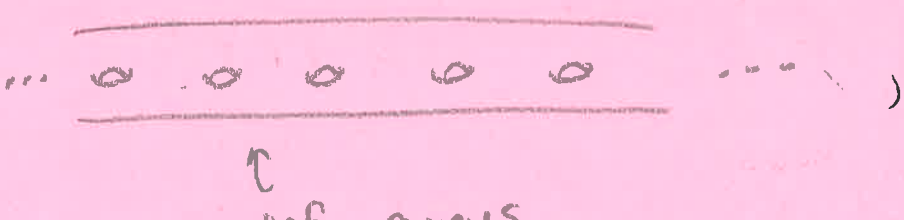
surface = connected, orientable 2-mfld w/o boundary

surface Σ is <u>finite-type</u> if $\pi_1(\Sigma)$ is fin. gen.

- classified by # genus & # punctures

surface Σ is <u>∞-type</u> if $\pi_1(\Sigma)$ is not fin. gen.

<u>Ex</u>:



, $S^2 \setminus 2^{\mathbb{N}}$

Def: $\mathrm{Map}(\Sigma) = \mathrm{Homeo}_+(\Sigma) / \mathrm{Homeo}_0(\Sigma)$

orientation-pres. homeos w/ cpt-open topology

connected component of identity

is a quotient of a Polish gp by a closed normal subgp $\Longrightarrow$ ~~so~~ a Polish group

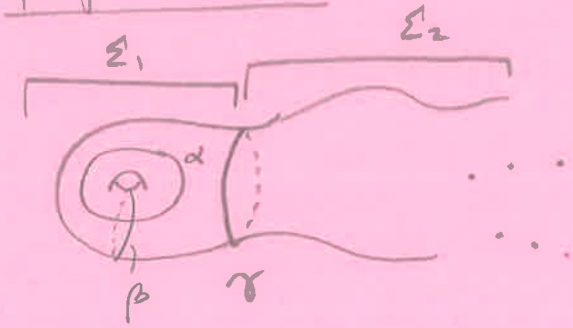
If Σ is finite-type $\Longrightarrow$ $\mathrm{Map}(\Sigma)$ is ctbl discrete

If Σ is ∞-type $\Longrightarrow$ $\mathrm{Map}(\Sigma)$ is non-locally cpt. & non-archimedean

$\mathrm{Map}(\Sigma) = \mathrm{Homeo}_+(\Sigma) / \sim$ ← isotopy equivalence relation

Thm (Domat - I. - Shinko) Σ ∞-type.
$\mathrm{Map}(\Sigma)$ has non-metrizable Furstenberg boundary.

proof sketch:



For F a fin. set of isotopy classes of simple closed curves on Σ,

$$\mathcal{N}_F = \{ f \in \mathrm{Map}(\Sigma) : f(\gamma) = \gamma \;\; \forall \gamma \in F \}$$

nbhd basis at 1_G as F varies.

Fact: ① $H \leq G$ open

If $\Pi_{sp}(G)$ metr, then $\Pi_{sp}(H)$ metr.

② $G \to H$ cts group hom

If $\Pi_{sp}(G)$ metr, then $\Pi_{sp}(H)$ metr.

Claim: The map

$$q: \mathcal{N}_{\{\gamma\}} \to \mathrm{Map}(\Sigma_1)$$

given by restriction is a cts group hom.

- $\ker(q) = \mathcal{N}_{\{\gamma, \alpha, \beta\}}$
- $\mathrm{Map}(\Sigma_1)$ is ctbl discrete non-amenable

finite type surface

So $\Pi_{sp}(\mathrm{Map}(\Sigma_1))$ non-metr $\Rightarrow \Pi_{sp}(\mathcal{N}_{\{\gamma\}})$ non-metr

$\Rightarrow \Pi_{sp}(\mathrm{Map}(\Sigma))$ non-metr.

The thm strengthens a result of Long: $\mathrm{Map}(\Sigma)$ is not amenable.